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"Criticality" in the counting function of prime numbers: Theory and application to nuclear magic numbers

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ABSTRACT

In this paper, considering critical phenomena and their phase transitions within the frame of the set of prime numbers, is attempted. Thus, the novel, theoretical and purely mathematical Model of Criticality based on Prime Numbers is introduced. This approach allows for the emergence of a parallelism between the physical concept of criticality and the corresponding concept in prime number theory. Based on this parallelism an application of the proposed model in determining the known magic numbers of Nuclear Physics is presented. This application introduces a physical meaning to the exceptional in properties set of the prime numbers and the corresponding prime number theory. Finally, going beyond the proposed model and its application, we suggest investigating other prime numbers as doubly magic numbers or candidate magic numbers for further experimental research.

1. Introduction

Stability and instability within the frame of criticality, as well as the phenomena of phase transitions and the emergence of critical point are concepts widely used in describing the world around us, including not only nature but also social and economic systems, to mention a few. These concepts are also found in pure mathematics and sciences, such as dynamical systems of differential equations [1–3], chaos theory [4], nonlinear systems [5,6] and in general in the modern theory of complexity. Parallelisms are often made between the mathematical characteristics of these concepts and the physical reality of critical phenomena and phase transitions [4,7].

In this work, we introduce the concepts of criticality and phase transitions in an exceptional and purely mathematical framework, that of the theory of prime numbers [8]. Motivated by our will to investigate and explore the existence of criticality in the set of prime numbers, we further move on to another goal, exploring possible applications of this concept in real physical systems. As a result, we introduce the theoretical and purely mathematical **M**odel of **Criticality** based on **P**rime **N**umbers (**MCPN**). Additionally, we apply this model on the magic numbers of Nuclear Physics.

The magic numbers of Nuclear Physics are the numbers of protons and/or neutrons that build the surprisingly stable nuclei. The property of stability is an especially important feature, if one considers that instability rather than stability characterizes isotopes. The phenomenon of this particular stability of magic numbers has been explained in Nuclear Physics through the model of shells [9,10]. In this paper, we attempt an interpretation of the existence of magic numbers by utilizing the introduced hereby **M**odel of **Criticality** based on **P**rime **N**umbers (**MCPN**). This would allow us to describe critical phenomena in nature in terms of prime number theory. Also, this would reveal, unknown of correspondences between prime number properties and a very important field, appearing everywhere in nature such as the critical phenomena.

The proposed parallelism between the MCPN and the magic numbers of nuclear physics motivates us to further search for other magic numbers, beyond the ones known so far or to extend to the notion of the doubly magic numbers. Finally, it should be mentioned that the application of the MCPN in the case of magic numbers, suggests a success for the theory of prime numbers, further gaining a physical content. One could say that it is a "hidden" description of the critical phenomena in nature.

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2. Gaps in the counting function π (x) of prime numbers

In the prime number theory, one may define the counting function (x), which counts the number of primes up to x. For this function, the definition is simply:

$$\pi(x) = \sum_{p\epsilon P} u(x-p), \tag{1}$$

where p is the set of primes numbers and $u(x)$ the Heaviside function, which is defined in the following relation: $u(x) = \begin{cases} 1\ for\ x \geq 0 \\ 0\ for\ x < 0 \end{cases}$. Imposing a gap $g = 2, 3, 4, 5\ldots, g \in Z$ on the set of prime numbers, one defines the following counting function:

$$\pi_g(x) = \sum_{p\epsilon P} u(x-(p-g)); g = 2, 3, 4\ldots \in Z. \tag{2}$$

We now create an arrangement of levels, where the number of each level is the position of the first number p in the series of prime numbers, until p becomes equal to x. For example, for the first 10 prime numbers we have: 1(2), 2(3), 3(5), 4(7), 5(11), 6(13), 7(17), 8(19), 9(23), 10(29), where 1, 2, 3, …, 10 stand for the position of prime number in the prime number series and inside the parenthesis appears the corresponding prime number value. From Eq. (2) one can find the counting number $\pi_g(x)$. As an example, we consider a gap $g = 7$, then we find for every level the corresponding counting number: 0, 0, 0, 0, 2, 3, 4, 5, 6, 8. We name these numbers, occupation numbers on their corresponding levels 1, 2, …, 10. This way, one may treat the prime numbers as if they were particles placed on levels; something that reminds us of physics, especially atomic and nuclear physics.

In the framework "logic of levels", one may also consider a known phenomenon which can be described in levels, that of the fine structure phenomenon, where a level is divided in two levels. A procedure like this may be described as:

$$\pi_g^{(-a)} = \sum_{p\epsilon P} u(x-(p-(g-a))) \tag{3}$$

and

$$\pi_g^{(+a)} = \sum_{p\epsilon P} u(x-(p-(g+a))) \tag{4}$$

where $a \in (0, 1)$.

Then the count functions $\pi_g^{(-a)}$, $\pi_g^{(+a)}$, provided by Eqs. (3) and (4), are the corresponding to occupation numbers of the prime numbers $p \leq x$ corresponding to these levels. In Fig. 1a an example for the first 15 prime numbers (1, 2, 3, …,15) and (a) a gap $g = 12$ and (b) a gap $g = 7$, is presented, taking into account that $a \in (0, 1)$.

Inspired by existing concepts coming from physics, we expand our syllogism considering the creation of dipoles (pairs) between "particles" that belong to the above defined populations $\pi_g^{(-a)}, \pi_g^{(+a)}$, for each level for the dipole $(-a, +a)$. Such a mathematical formulation could represent a a two-way spin physical system with a down (-1) and up $(+1)$ direction. It is apparent that there are gaps for which we get images such as that in Fig. 1b, i.e., for each level it holds that:

$$\pi_g^{(-a)} = \pi_g^{(+a)} \tag{5}$$

To distinguish the specific values of the gaps that fulfill Eq. (5), we call them "*resonances*", a word borrowed from physics. It must be reminded that the classical phenomenon of resonance is obtained when the frequency of the stimulator becomes equal to the eigen-frequency.

Another concept to use, coming from the field of physics, is the concept of shells. These are structures larger than levels; for example, in nuclear physics energy shells contain energy levels. In this case the occupation number in the proton-neutron shells is calculated cumulatively, by summing the occupation numbers of the levels contained.

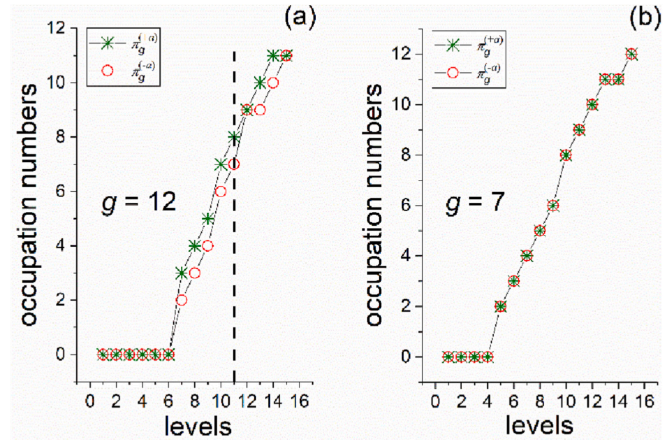

Fig. 1. Red points refer to $\pi_g^{(-a)}$ and green points to $\pi_g^{(+a)}$. (a) The occupation numbers corresponding to the level for $g = 12$. The vertical line shows these occupation numbers in level equal to 11 (i.e in 11-th prime number in series). (b) The occupation numbers correspond to the levels for $g = 7$. Now the occupation numbers become equal for every level of the fine structure. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Correspondingly, one may define shells in the case of prime numbers as those where the cumulative occupation numbers in each level are calculated by adding to the occupation number of the level and the corresponding occupation numbers of the previous levels. In the rest, we will refer to the cumulative case in prime numbers.

To quantify the difference between the gaps that produce the resonances of Eq. (5), as well as all the other gaps that do not produce resonances, we define the following quantity:

$$Q = \left(\frac{N^{(0)}}{N}\right) \bullet 100\%, \tag{6}$$

where N is the population of the considered prime numbers, or N is the number of the pairs $(-a, +a)$, and $N^{(0)}$ is the number of pairs where the Eq. (5) is valid. In Fig. 2, a graph showing the escalation of quantity Q vs. the value of 60 gaps, for the first $N = 168$ prime numbers that cover the first thousand of the integers, appears.

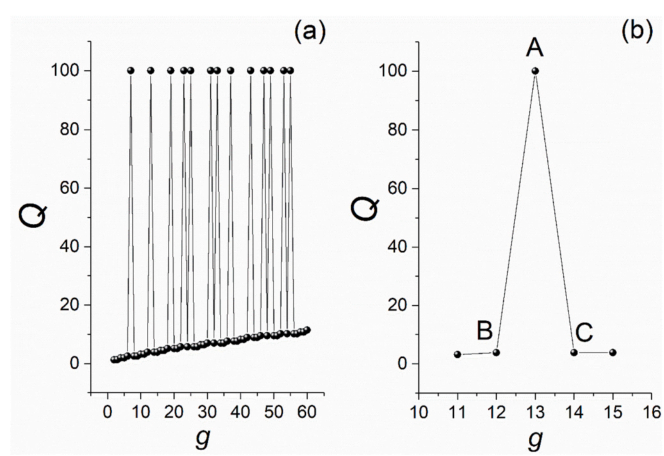

Fig. 2. (a) The quantity Q vs. the gap $(g) = 2, \ldots, 60$, considering the first 168 prime numbers. (b) Zoom in a spike. Point A ($g = 13$, $Q = 100\%$), point B (12, 3.8%) and C (14, 3.8%).

3. Parallelism between "criticality" in prime numbers and phase transitions in physics

Critical phenomena and phase transitions appear to be universal framework embracing phenomena in nature and furthermore. A short and very good presentation of the theory of criticality can be found in [11]. In brief, the critical point or critical state for dynamical systems is the point where two phases are separated, and this is the key concept. These two phases are the phase of high symmetry and the phase of broken symmetry. A critical point appears just before the transition from one phase to another and the procedure is controlled by changing the value of a parameter of the system (control parameter). As an example, we mention ferromagnetic materials. In these materials the evolution of the mean magnetization is monitored as the temperature is changed, thus, playing the role of the control parameter. The mean magnetization is characterized as an order parameter and has the property of being zero (without external magnetic field) in the symmetric phase, while it has a non-zero value at critical point in the phase of the broken symmetry. In numerical models such as the Ising model, the magnetic momentum (spin) of the material is allowed to have only two possible orientations, the upward direction (+1) and the downward direction (−1). Thus, the order parameter, i.e., the average magnetization, is defined as the component of these two directions. At the critical point the number of positive spins is exactly equal to the number of negative spins, therefore, the mean magnetization is zero.

Considering the described in the previous section mathematical model, one may correspond to the possible orientations the quantities $\pi_g^{(-a)}$ and $\pi_g^{(+a)}$ in the occupation numbers of the prime numbers. Also, the in the position of the critical point, one may correspond the resonance in the prime numbers, where $\pi_g^{(-a)} = \pi_g^{(+a)}$. Additionally, the order parameter ϕ may be corresponded to the quantity $\phi = 100 - Q$, where in the critical point (resonances) becomes 0, while the control parameter would be recognized the gap g.

In a second order phase transition the phase diagram, i.e., a graph illustrating the dependence of order parameter on the control parameter, has the form appearing in Fig. 3a [11,12]. Comparing the two graphs of Fig. 3 we can see a correspondence between the critical point of phase transition and the resonance gap of prime numbers. This correspondence is consistent to the term "resonance" that we opted to use, considering a recent work about the critical point of a phase transition is a state of resonance [13].

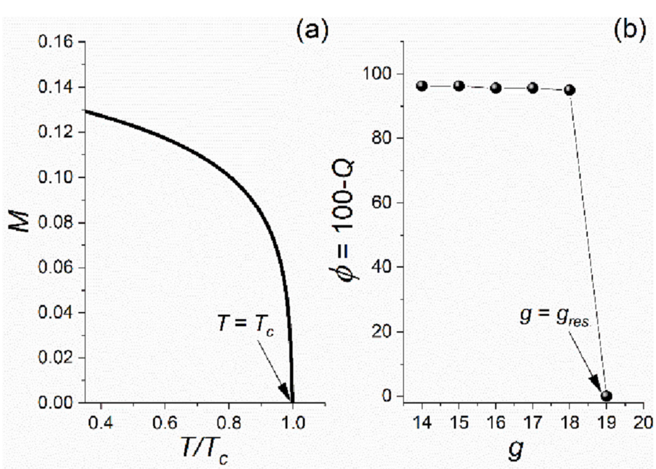

Fig. 3. (a) The phase diagram for a thermal system undergoing a second order phase transition. In this diagram magnetization M is the order parameter, while temperature T is the control parameter. The external magnetic field is zero. The arrow shows the position of critical point where $M = 0$. (b) The quantity ϕ, as an order parameter and the gaps g as the control parameter in case of the prime number representation. The arrow shows the position of resonance gap where $\phi = 0$. The points are the points between the successive resonances according to Fig. 2a.

The basic phase transitions which take place in nature are the first and second order phase transitions. The transition from the symmetrical phase to the phase of the broken symmetry is accomplished according to the description of the mean field theory based on the Landau free energy [11]. According to the spontaneous symmetry breaking (SSB) scenario in the case of the second order phase transition, the critical point separating a stable point in the symmetric phase is converted into an unstable point in the phase of the broken symmetry, further granting its position to two stable states.

In Fig. 4 a typical SSB phenomenon is illustrated [11,14]. As shown in this, the stable points are degenerate, since they have the same free energy. This means the two stable points are perfectly equivalent and that the system will eventually settle in one of these two. Attempting to perform a correspondence between this diagram of the typical second order phase transition (Fig. 4) and the one appearing in Fig. 2b of the prime numbers we can recognize in Fig. 2a the point A as an "unstable point" at resonance gap position $g = 13$ and therefore the points C, B as "stable points" at positions 13 ± 1 respectively. In addition, we see that also the points B, C are degenerate because their coordinates have the same value for the quantity Q.

The other type of phase transition that occurs in nature is the first order phase transition where the free energy diagram with the order parameter shows a triple degeneration as shown in Fig. 5.

In Fig. 2a for the case of the prime numbers twin resonances also appear and they have the form of Fig. 6, appearing additionally to the resonances for the case of Fig. 2b.

According to the presented syllogism in this section, it is apparent that the introduction of the concept of critical phenomena into prime numbers is performed at the level of correspondences, similarities, and parallelisms; parallel descriptions between quantities coming from mathematical theory, such as the prime numbers, and physical reality such as critical phenomena.

Summarizing all the above thinking pathway, these correspondences are presented in a systematic and illustrative way in Table 1. In this table all the typical quantities appearing in phase transitions are corresponded one to one to quantities of the set of prime numbers. As seen in this table the elements describing phase transitions corresponded to properties prime numbers are magnetic moments (spins), order parameter, control parameter, symmetries, as well as other characteristics of phase transitions like critical point and stable vacua. Additionally, the phase transitions themselves correspond to prime numbers'

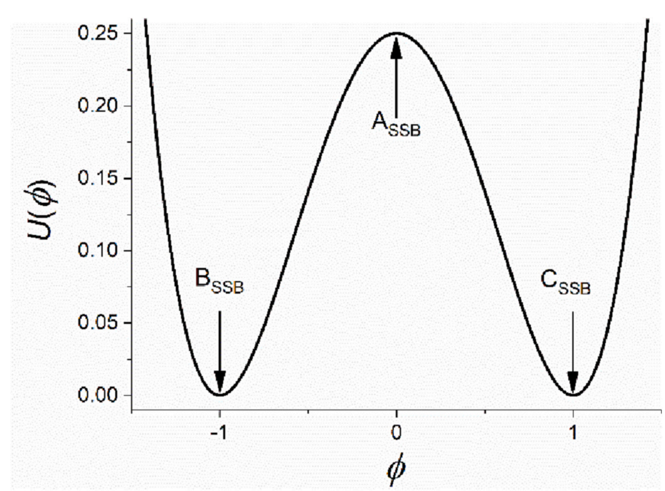

Fig. 4. The Landau free energy $U(\phi)$ vs. the order parameter ϕ in a typical SSB phenomenon during a second order phase transition. The critical point A is an unstable point, which gives its position in the broken symmetry phase to the two stable points B and C.

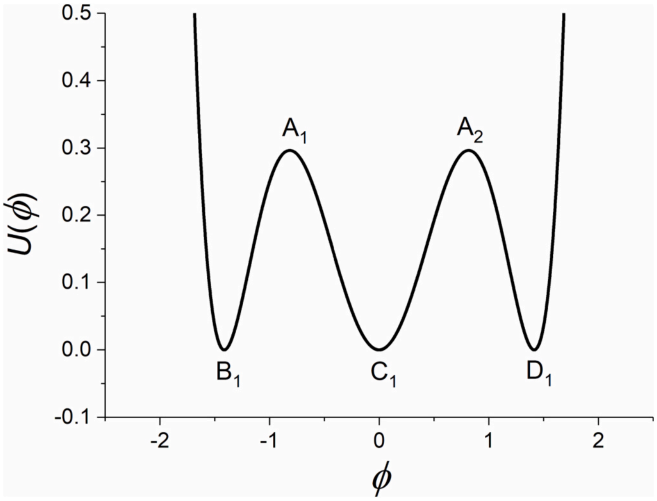

Fig. 5. The Landau free energy $U(\phi)$ vs. the order parameter ϕ in the case of first order phase transition. A triple degeneration for the order parameter is shown.

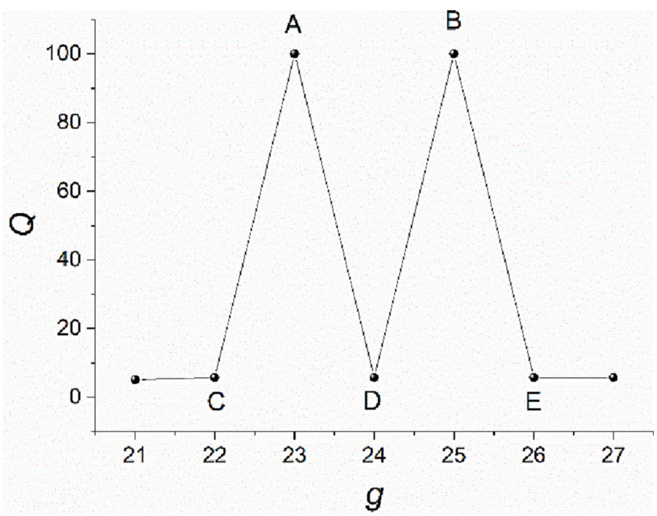

Fig. 6. A segment of the graph appearing in Fig. 2a, which resembles a twin resonance. For $g = 23, 25$ unstable critical points appear. These points correspond to the unstable critical points $A1$ and $A2$ in Fig. 5. The stable points C, D, E at the positions 22, 24, 26, correspond to the same value of Q and therefore are degenerate.

Table 1
Correspondences between thermal critical phenomena (and phase transitions) and Counting Functions with gap in prime number theory.

Thermal critical phenomena and phase transitions	Counting function with gap in prime number theory
Ising total magnetic moments, $\text{Spin}^{(-)}$, $\text{Spin}^{(+)}$	Occupation number $\pi_g^{(-a)}$, $\pi_g^{(+a)}$
Critical point: $Spin^{(-)} = Spin^{(+)}$	"critical point", $\pi_g^{(-a)} = \pi_g^{(+a)}$
Critical point at $T = T_c$	"critical point" at $g = g_{resonance}$
Control parameter: Temperature	Control parameter: gap
Order parameter: Magnetization M	Order parameter: $\Phi = 100 - Q$
Stable potential vacua are in symmetrical positions of critical point	"Stable" gaps in symmetrical positions of $g_{resonance}$
Degenerated stable vacua	Degenerated "stable" gaps
Second order phase transition	Single resonances
First order phase transition	Twin resonances

resonances (being consistent to [13], where phase transitions are, in fact, resonance phenomena). This way a physical meaning for prime numbers emerges.

Beginning from physics and the concepts within the frame regarding critical phenomena, we arrived at the corresponding concepts in the set of prime numbers. Conversely, one could use now these concepts in the field of prime numbers to identify physical systems that are in critical state. Then we treat the concepts in prime numbers like a model, for which we coin the name "Model Criticality based on Prime Numbers" (**MCPN**).

A confirmation of the above could be made if characteristic numbers referring to stable physical systems are identified with the values of the "stable" gaps for the counting functions of prime numbers, via the MCPN. Such characteristic physical numbers are the populations of protons or neutrons in those nuclei which exhibit exceptional stability. These numbers are the magic numbers of nuclear physics.

4. Application of the MCPN on magic numbers of nuclear physics

The magic numbers of nuclear physics are the numbers of protons (Z) or the number of neutrons (N) of the nuclei that have the property of being extremely stable. The seven most widely recognized magic numbers are **2**, **8**, **20**, **28**, **50**, **82**, and **126** [15,16]. Further, predicted magic numbers are **114**, **122**, **124**, and **164** for protons [17–20]. Additional calculations predict, along with **184**, **228** also **308** for neutrons [20,21]. In September 2019 an international research team stated that the number **34** for neutrons is a magic number [22]. Goeppert-Mayer and other physicists went on to explain this phenomenon based on the nuclear shell model, in which protons and neutrons fill a nucleus in energy shells, or orbital, akin to the layers of an onion.

Nuclei which have neutron numbers and proton numbers each equal to one of the magic numbers are called "doubly magic" and are especially stable against decay [23]. Such a nucleus is the isotope of Nickel with $A = N + Z = 50 + 28 = 78$ [24].

The fact that in recent years there has been significant research on the issue of magic numbers, as proved by mentioned works, means that the issue of magic numbers is not closed. On the contrary, intense experimental efforts still go on to locate other magic numbers, since the issue of finding new stable nuclei that are unsusceptible to nuclear decay will always be a challenge in nuclear physics.

It is known from the shells model of nucleus, that magic numbers correspond to the greatest gaps in energy between shells, providing extra stability to nucleus, which is completely filled in, with those shells [25,26].

Getting back to the proposed in this work model, the procedure provided by the MCPN as this is applied in the case of calculating the magic numbers of nuclear physics, is presented in the form of steps:

- We locate the positions $g_{resonance}$ of the resonance gaps.

a) Then we take as stable positions only those gaps provided by the theory of criticality, i.e. the breaking of the symmetry. These are the ones that have a gap value of $g_{resonance} \pm 1$. From this procedure we exclude the magic number 2, because it has an odd neighbor the number 1, which, however, does not correspond to a gap because $g > 2$. In Table 2 we present the results of the application of the MCPN for the classic magic numbers 8, 20, 28, 50, 82, 126 and the new ones 34, 114, 122, 124, 164, 184, 228, 308. In column 3 the numbers of the stable gaps of the prime numbers that are identical with the known magic numbers (for example 8, 20, 50, 82, …etc.), are presented.

- It appears that the magic number covers the one of the two stable degenerate gap positions for single resonances or up to two from three positions, for twin resonances.

Table 2
"Stable gaps and Magic Numbers".

No	Resonance gap/kind	Stable gaps ±1	Magic numbers
1	7/single	6,8	8
2	19/single	18,20	20
3			Semi-magic 28
4	47,49/twins	46,48,50	50
5	83,85/twins	82,84,86	82
6	127/single	126,128	126
7	31,33/twins	30,32,34	34
8	113/single	112,114	114
9	121,123/twins	120,122,124	122,124
10	163/single	162,164	164
11	183/single	182,184	184
12	229/single	228,230	228
13	307/single	306,308	308

- The gap 28 is very close to the stable gap 26, which belongs as a *C*-point to the resonance gap 25. So according to our model it is not a magic number. Indeed, in the first series of magic numbers number 28 was not included [27]. In [28] there is a discussion that less-pronounced effects are also observed for 28, 14, 40. These numbers are known as semi-magic numbers [29], due to the special stability of their nucleus, thus they are less marked than a magic number. Therefore, our model can successfully discern this slight variation in the stability of the magic number 28.

In the following we highlight the main similarities between the MCPN and magic numbers.

- As we see the concepts of energy gap and nucleus stability are the most important in nuclear magic numbers theory just like gaps and stability in counting function of prime number theory.
- In the model of the shells of the nuclei, the occupation numbers of proton and neutrons are filled according to the cumulative application of the Pauli exclusion principle on the sub-shells. In prime numbers, the creation of occupation numbers at the levels included to a gap is also performed through Eqs. (3) and (4) with a cumulative way too.
- The "coincidence" that the magic numbers are the same as the gap numbers in Table 2, exists because we have limited the possible positions of stability only to those predicted by the theory of critical phenomena and phase transitions in our model MCPN. This important conclusion leads us to think that the phenomenon of magic numbers obeys the mechanisms of first and second order phase transitions. The order of the phase transition could be predicted by checking whether the magic number is a neighbor to simple or twin resonances. Thus, in our effort to find extended connections between prime numbers and magic numbers, we realized that the critical phenomena and phase transition could be such a field.
- Phase transition has also been studied in the field of magic numbers, and both cases of first and second order as well as criticality have been studied in nuclear phenomena [30–33], in particular, on issues concerning the shape of nuclei and their stability.

In [34] the curve of the first order phase transition (Fig. 5) corresponds to the critical point of the spherical-to-deformed transition, in the framework of the study of the transition of the shape of the nucleus with magic numbers through the phase transition. This field is an open and challenging topic for further investigation in the framework of novel model introduced hereby (MCPN).

5. Discussion and concluding remarks

Beginning from the fact that the number of magic numbers is relatively small and looking at Fig. 2a, it is apparent that there are resonances that do not correspond to the magic numbers found till now. Initially, we must observe (Fig. 2a) that as the gap in the prime numbers increases, the level of degenerate stable gaps increases too. This means that the stable vacuum in Landau free energy becomes shallower or otherwise the stability decreases. That is, magic numbers are no longer so "magic", confirming a well-known result in nuclear physics that the heavy nuclei show less stability and from some value of magic numbers and on, we can no longer talk about them.

Of course, the initial question of the correspondence between the population of known magic numbers and the population of resonances remains. It is possible to consider at least three views. One view is that nature does not invest on other resonances in magic numbers but perhaps on other natural phenomena that refer to critical state. The second view is that these extra magic numbers, corresponding to the excess of resonances, have not been found experimentally. Third view is to correspond the excess of resonances to magic numbers using the concept of the doubly magic numbers.

We will follow what we said before about the decrease of stability as the gap increases and we will go that way until gap 94. Then appears an excess of single resonances for 13, 37, 43, 67, 79 and the twin resonances 23–25, 53–55, 61–63, 73–75, 91–93. Thus, the degenerate positions for the single resonances according to MCPN are (12, 14), (36, 38), (42, 44), (66, 68), (78, 80). Then the corresponding candidate magic number is one of the two numbers inside the parentheses. For the twin resonances the most probable positions are the positions *D* (Fig. 6), which are stable states of two resonances simultaneously. Then the emerging candidate magic numbers are 24, 54, 62, 74, 92.

Using the concept of doubly magic numbers, most of the candidate magic numbers are covered by various combinations of existing magic numbers and candidates. We present various examples based on doubly magic number:

a) Number 78 is already a doubly magic number (section 4).
b) The existing magic number 20 can be considered as doubly magic number by the already known 8 and the candidate 12 ($20 = 8 + 12$).
c) The candidate magic number 42 with the existing magic number 50 gives the candidate 92 as a doubly magic number.
d) The candidate magic number 24 with the existing magic number 50 gives the candidate 74 as a doubly magic number.
e) The candidate 54 and 38 gives the candidate 92 as a doubly magic and so on.

Concluding, we should mention that in this paper following a parallelism between the concept of criticality in nature and nonlinear mathematical theories, we transfer the concept of criticality to the set of prime numbers through the counting function. We further developed and introduced the Model of Criticality in Prime Numbers (MCPN), which we applied to calculate the magic numbers in nuclear physics, based on two common basic concepts of our model and magic numbers, namely stability and gaps. Finally, we concluded that the known magic numbers are determined by MCPN. In addition, we suggested for further research investigating the possible existence of other magic numbers, bearing mainly the property of doubly magic numbers.

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CRediT authorship contribution statement

Y.F.C., S.G.S., P.P., M.P.H. and S.M.P. conceived the paper's idea; Y.F.C., P.P. and S.G.S. devised the methodology and coordinated the work; Y.F.C., P.P. and N-L.M. performed the experimental part, Y.F.C., S.G.S., P.P, M.P.H and S.M.P. analyzed the results; All authors participated in writing and reviewing the manuscript.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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